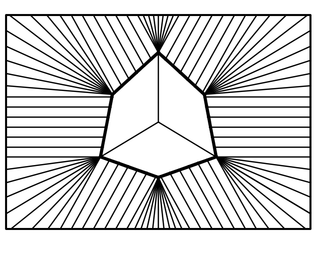
PLAXFLOW

Scientific Manual

Version 1.4



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1 INTRODUCTION

Groundwater flow is an important issue in many engineering fields such as geotechnical, environmental, agricultural and hydrological engineering. This subject has received a lot of attention in the literature and many excellent books and contributions in scientific journals are available.

Therefore, in this part of the manual only some theoretical background is given. Attention is focused on the material models and the boundary conditions involved in PLAXFLOW, as these particularly demonstrate the advantages of PLAXFLOW in performing groundwater flow analysis for a variety of engineering applications. These applications may include confined and unconfined flow, infiltration, precipitation, sinks and sources. Such conditions may be applied for saturated and unsaturated conditions under both steady-state and transient flow.

2 GROUNDWATER FLOW THEORY

In this chapter we will review the theory of groundwater flow as used in PLAXFLOW. In addition to a general description of groundwater flow, attention is focused on the finite element formulation.

2.1 BASIC EQUATIONS

Flow in a porous medium can be described by Darcy's law. Considering flow in a vertical x-y-plane the following equations apply:

$$q_x = -k_x \frac{\partial \phi}{\partial x} \qquad q_y = -k_y \frac{\partial \phi}{\partial y} \tag{2.1}$$

The equations express that the specific discharge, q, follows from the effective permeability, K (main components k_x and k_y), and the gradient of the groundwater head. The head, ϕ, is defined as follows:

$$\phi = y - \frac{p}{\gamma_w} \qquad \phi_p = -\frac{p}{\gamma_w} \tag{2.2}$$

where y is the vertical position, p is the stress in the pore fluid (negative for compressive pressure) and γ_w is the unit weight of the pore fluid. In addition to the head, a pore pressure head, ϕ_p (positive for compression), is defined that will be used in the definition of the material models. For transient flow the continuity condition applies:

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + c\frac{\partial \phi}{\partial t} = Q \tag{2.3}$$

Eq. (2.3) expresses that there is no net inflow or outflow in an elementary area per unit of time. Figure 2.1 illustrates the steady state part in absence of the source term Q.

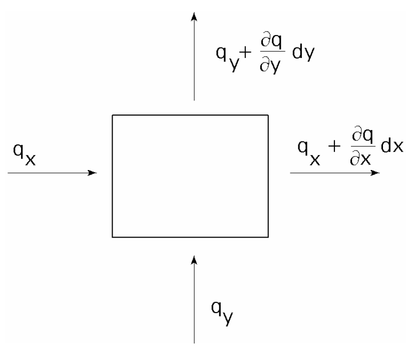

Figure 2.1 Illustration of the continuity condition

The transient part relates the change in head to a change in volumetric water content of the elementary volume via the effective capacity c. Opposite to saturated flow the effective capacity and effective permeability depend on the water content and the capillary pressures per soil type.

The effective capacity, c, is often expressed as:

$$c = c_{sat} + n \frac{dS(\phi_p)}{d\phi_p} \tag{2.4}$$

The effective permeability tensor, K, can then be related to the saturation as:

$$K = k_{rel}(S)\ K_{sat} \tag{2.5}$$

Both saturation, S, and relative permeability k_{rel}, are written as single valued functional relations in PLAXFLOW.

2.2 MATERIAL MODELS

The modelling of unsaturated flow is mostly based on a Van Genuchten material description. According to this model the saturation depending on effective pressure head reads:

$$S(\phi_p) = S_{residu} + (S_{sat} - S_{residu})\left(1 + \left(g_a \left|\phi_p\right|\right)^{g_n}\right)^{\left(\frac{1-g_n}{g_n}\right)} \tag{2.6}$$

Van Genuchten assumes a residual saturation S_{res} which describes a part of the fluid that remains in the pores even at high suction heads. In general at saturated conditions the pores will not be completely filled with water as air can get trapped and the saturation in this situation, S_{sat}, will be less than one. The remaining parameters g_a, g_l and g_n have to be measured for a specific material. Relative permeability is related to the saturation via the effective saturation. The effective saturation S_e is expressed as:

$$S_e = \frac{S - S_{residu}}{S_{sat} - S_{residu}} \tag{2.7}$$

The relative permeability according to Van Genuchten now reads:

$$k_{rel}(S) = (S_e)^{g_l} \left(1 - \left(1 - S_e^{\left(\frac{g_n}{g_n - 1}\right)} \right)^{\left(\frac{g_n - 1}{g_n}\right)} \right)^2 \tag{2.8}$$

Figure 2.2 and Figure 2.3 present the Van Genuchten relations for a sandy material with parameters S_{sat} = 1.0, S_{residu} = 0.027, g_a = 2.24 m^{-1}, g_l = 0.0 and g_n = 2.286 graphically.

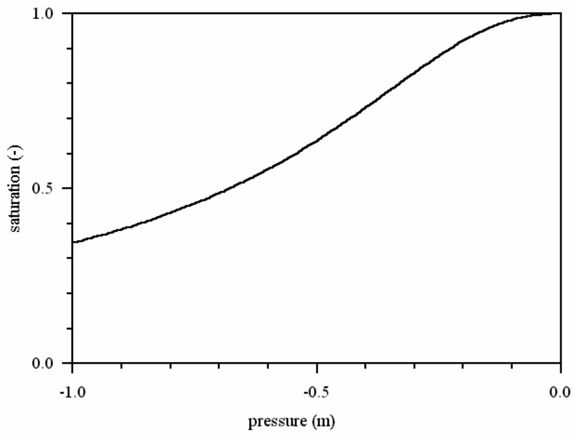

Figure 2.2 Van Genuchten pressure-saturation

Note that using the expression for saturation, relative permeability can be related to the effective pressure directly.

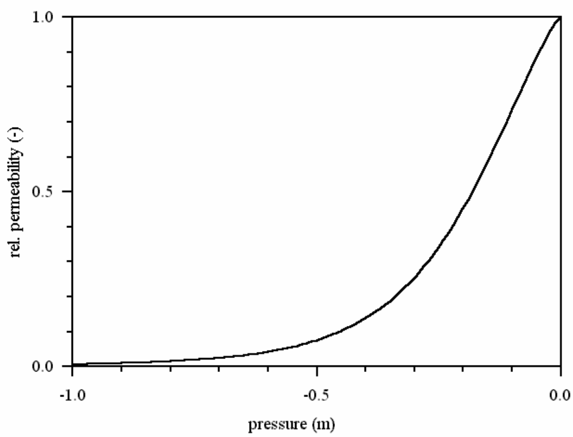

Figure 2.3 Van Genuchten pressure-relative permeability

As an alternative, the PLAXFLOW program supports a linearized Van Genuchten model for which the approximate Van Genuchten parameters can be derived. According to this concept saturation relates to the pore pressure head as:

$$S(\phi_p) = \begin{cases} 1 & if \quad \phi_p \geq 0 \\ 1 + \dfrac{\phi_p}{|\phi_{ps}|} & if \quad \phi_{ps} < \phi_p < 0 \\ 0 & if \quad \phi_p \leq \phi_{ps} \end{cases} \tag{2.9}$$

The variable ϕ_{ps} is a material dependent pressure head which specifies the extent of the unsaturated zone under hydrostatic conditions. Below this threshold value the saturation is assumed to be zero. For saturated conditions the degree of saturation equals one. The relation between relative permeability and pressure head is written as:

$$k_{rel}(\phi_p) = \begin{cases} 1 & if \quad \phi_p \geq 0 \\ 10^{\frac{4\phi_p}{|\phi_{pk}|}} & if \quad \phi_{pk} < \phi_p < 0 \\ 10^{-4} & if \quad \phi_p \leq \phi_{pk} \end{cases} \tag{2.10}$$

According to this formulation the permeability in the transition zone is described as a log-linear relation of pressure head where ϕ_{pk} is the pressure head at which the relative permeability is reduced to 10^{-4}. The permeability remains constant for higher values of the pressure head. Under saturated conditions the relative permeability equals one and the effective permeability is equal to the saturated permeability which is assumed to be constant.

The input parameters of the “approximate Van Genuchten model” are derived from the classical Van Genuchten model. These parameters are translated into approximately equivalent process parameters for the numerically more robust linearized model. For ϕ_{ps} the translation is as follows:

$$\phi_{ps} = \frac{1}{S_{\phi_p = -1.0\text{m}} - S_{sat}} \tag{2.11}$$

The parameter ϕ_{pk} is set equal to the pressure head at which the relative permeability according to Van Genuchten is 10^{-2}, with a lower limit of -0.5 m. Figure 2.4 presents the functional relation between pressure and saturation according to the approximate Van Genuchten model using ϕ_{ps} = 1.48 m. The corresponding pressure-relative saturation relation ϕ_{pk} = 1.15 m is given in Figure 2.5.

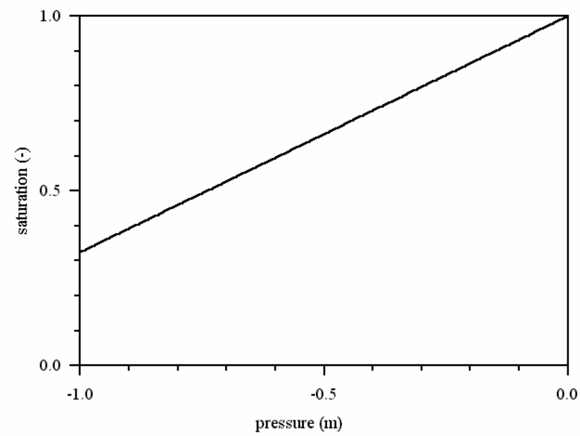

Figure 2.4 Approximate Van Genuchten pressure-saturation

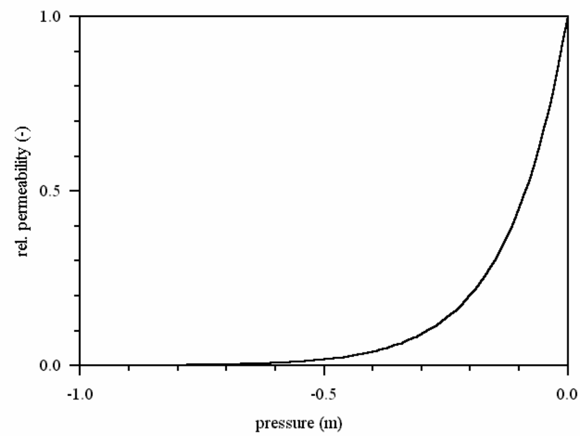

Figure 2.5 Approximate Van Genuchten pressure-relative permeability

The groundwater flow problem now has to be completed by boundary conditions and initial conditions. The next paragraph will present the set of conditions supported by PLAXFLOW.

2.3 BOUNDARY CONDITIONS

Closed boundary conditions specify a zero Darcy flux over the boundary as

$$q_x n_x + q_y n_y = 0 \tag{2.12}$$

where n_x and n_y are the outward pointing normal vector components on the boundary.

Inflow over a boundary is set by a prescribed recharge value $\bar{q}$ and reads

$$q_x n_x + q_y n_y = -\bar{q} \tag{2.13}$$

This indicates that the Darcy flux vector and the normal vector on the boundary are pointing in opposite directions.

Outflow boundary conditions are handled in a similar way, discharge $\bar{q}$ equals

$$q_x n_x + q_y n_y = \bar{q} \tag{2.14}$$

For outflow the direction of the Darcy flux should equal the direction of the normal on the boundary.

For prescribed head boundaries the value of the head $\bar{\phi}$ is imposed as

$$\phi = \bar{\phi} \tag{2.15}$$

Alternatively prescribed pressure conditions can be given. Overtopping conditions for example can be formulated as prescribed pressure boundaries.

$$p = 0 \tag{2.16}$$

These conditions directly relate to a prescribed head boundary condition and are implemented as such.

Infiltration poses a more complex mixed boundary condition. An inflow value $\bar{q}$ may depend on time and as in nature the amount of inflow is limited by the capacity of the soil. If the precipitation rate exceeds this capacity, ponding takes place at a depth $\bar{\phi}_{p,\max}$ and the boundary condition switches from inflow to prescribed head. As soon as the soil capacity meets the infiltration rate the condition switches back.

$$\begin{cases} \phi = z + \bar{\phi}_{p,\max} & \text{if} \quad ponding \\ q_x n_x + q_y n_y = -\bar{q} & \text{if} \quad \phi < z + \phi_{p,\max} \cap \phi > z + \phi_{p,\min} \\ \phi = z + \bar{\phi}_{p,\min} & \text{if} \quad drying \end{cases} \tag{2.17}$$

This boundary condition simulates evaporation for negative values of $\overline{q}$. The outflow boundary condition is limited by a minimum head $\overline{\phi}_{p,\min}$ to ensure numerical stability.

The water line option generates phreatic/seepage conditions by default. An external head $\overline{\phi}$ is prescribed on the part of the boundary beneath the water line, seepage or free conditions are applied to the rest of the line. The phreatic/seepage condition reads

$$\begin{cases} \phi = \overline{\phi} & if \quad \overline{\phi} \geq z \\ q_x n_x + q_y n_y = 0 & if \quad \overline{\phi} < z \quad \cap \quad \phi < z \\ \phi = z & if \quad \overline{\phi} < z \quad \cap \quad q_x n_x + q_y n_y > 0 \end{cases} \tag{2.18}$$

The seepage condition only allows for outflow of groundwater at atmospheric pressure. For unsaturated conditions at the boundary the boundary is closed.

Alternatively a water line may generate a phreatic/closed condition if the upper part of the line is replaced by closed conditions. This condition is written as

$$\begin{cases} \phi = \overline{\phi} & if \quad \overline{\phi} \geq z \\ q_x n_x + q_y n_y = 0 & if \quad \overline{\phi} < z \end{cases} \tag{2.19}$$

The external head $\overline{\phi}$ may vary in a time dependent way, however the part that remains closed is derived from the initial setting.

Inside the domain wells are modelled as source terms, where $\overline{Q}$ specifies the inflowing flux per meter.

$$Q = \overline{Q} \tag{2.20}$$

As the source term in the governing equation simulates water flowing in the system, the source term is positive for a recharge well.

A discharge rate $\overline{Q}$ simulates an amount of water leaving the domain

$$Q = -\overline{Q} \tag{2.21}$$

The source term in the governing equation is negative for a discharge well.

Drains are handled as seepage boundaries. However, drains may be located inside the domain as well. The condition is written as

$$\begin{cases} \phi = z & if \quad Q < 0 \\ Q = 0 & if \quad \phi < z \end{cases} \tag{2.22}$$

A drain permits water leaving the modelling domain at atmospheric pressure. The drain itself does not generate a resistance against flow.

Initial conditions are generated as a steady state solution for a problem with a given set of boundary conditions.

2.4 FINITE ELEMENT DISCRETISATION

The groundwater head in any position within an element (e) can be expressed in terms of nodal values:

$$\phi(\xi,\eta) = \underline{N}\,\underline{\phi}^e \tag{2.23}$$

where $\underline{N}$ is the vector with interpolation functions and ξ and η are the local coordinates within the element. For more information on the finite element theory please refer to the PLAXIS scientific manual or Bathe [2], Zienkiewicz [7]. According to Eq. (2.1) the specific discharge is based on the gradient of the groundwater head. This gradient can be determined by means of the $\underline{\underline{B}}$-matrix, which contains the spatial derivatives of the interpolation functions.

$$\underline{\underline{B}} = \begin{bmatrix} \dfrac{\partial \underline{N}}{\partial x} \\ \dfrac{\partial \underline{N}}{\partial y} \end{bmatrix} \tag{2.24}$$

In the numerical formulation the specific discharge, $\underline{q}$, is written as:

$$\underline{q} = -k_{rel}\,\underline{\underline{R}}\,\underline{\underline{B}}\,\phi^e \tag{2.25}$$

where:

$$\underline{q} = \begin{bmatrix} q_x \\ q_y \end{bmatrix} \qquad \text{and:} \qquad \underline{\underline{R}} = \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \tag{2.26}$$

From the specific discharges in the integration points, $\underline{q}$, the nodal discharges $\underline{Q}^e$ can be integrated according to:

$$Q^e = -\int \underline{\underline{B}}^T\,\underline{q}\;dV \tag{2.27}$$

in which $\underline{\underline{B}}^T$ is the transpose of the *B*-matrix. On the element level the following equations apply:

$$Q^e = \underline{\underline{K}}^e\,\underline{\phi}^e \qquad \text{with:} \qquad \underline{\underline{K}}^e = \int K^r\,\underline{\underline{B}}^T\,\underline{\underline{R}}\,\underline{\underline{B}}\;dV \tag{2.28}$$

and

$$\dot{\underline{Q}}^e = \underline{\underline{C}}^e \frac{d\underline{\phi}^e}{dt} \qquad \text{with:} \qquad \underline{\underline{C}}^e = \int \underline{N}^T \; c \, \underline{N} \; dV \tag{2.29}$$

for the transient part.

On a global level, contributions of all elements are added and boundary conditions (either on the groundwater head or on the recharge) are imposed. This results in a set of *n* equations with *n* unknowns:

$$\underline{Q} = \underline{\underline{K}}\underline{\phi} + \underline{\underline{C}} \frac{d\underline{\phi}}{dt} \tag{2.30}$$

in which $\underline{\underline{K}}$ is the global steady state flow matrix, $\underline{\underline{C}}$ is the global transient flow matrix and $\underline{Q}$ contains the prescribed recharges that are given by the boundary conditions.

Due to the unsaturated zone the set of equations is highly non-linear and a Picard scheme is used to solve the system of equations iteratively. The linear set is solved in incremental form using an implicit time stepping schema. For each iteration increments of the groundwater head are calculated from the unbalance in the nodal discharges and added to the active head. From the new distribution of groundwater heads the new specific discharges are calculated, which can again be integrated into nodal discharges. This process is continued until the norm of the unbalance vector, i.e. the error in the nodal discharges, is smaller than the tolerated error.

Interface elements are treated specially in groundwater calculations. The elements can be on or off. When the elements are switched on, there is a full coupling of the pore pressure degrees of freedom. When the interface elements are switched off, there is no flow from one side of the interface element to the other (impermeable screen).

3 REFERENCES

[1] Bakker K.J. (1989), Analysis of groundwater flow through revetments. Proc. 3rd International Symposium on Numerical Models in Geomechanics. Niagara Falls, Canada. pp. 367-374.

[2] Bathe K.J., Koshgoftaar M.R. (1979), Finite element free surface seepage analysis without mesh iteration. Int. J. Num. An. Meth Geo, Vol. 3, pp. 13-22.

[3] Desai C.S. (1976), Finite element residual schemes for unconfined flow. Int. J. Num. Meth. Eng., Vol. 10, pp. 1415-1418.

[4] Li G.C., Desai C.S. (1983), Stress and seepage analysis of earth dams. J. Geotechnical Eng., Vol. 109, No. 7, pp. 946-960.

[5] Riks E. (1979), An incremental approach to the solution of snapping and buckling problems. Int. J. Solids & Struct. Vol. 15, pp. 529-551.

[6] Van Genuchten. M. Th. (1980), A closed form equation for predicting the hydraulic function of unsaturated soils, Journal Soil Science Society of America, 44.

[7] Zienkiewicz O.C. (1967), The finite element method in structural and continuum mechanics. McGraw-Hill, London, UK.

APPENDIX A - SYMBOLS

$\underline{\underline{B}}$	:	Spatial derivatives of the shape functions
c (l/m)	:	Effective capacitance
c_{sat} (l/m)	:	Elastic compressibility
k_{rel}	:	Relative permeability function
K (m/day)	:	Effective permeability tensor
K_{sat} (m/day)	:	Saturated permeability tensor
$\underline{\underline{K}}$ (m/day)	:	Flow matrix
n	:	Porosity
$\underline{\underline{N}}$	:	Matrix shape functions
P (kN/m^2)	:	Pore pressure (negative for pressure)
q (m/day)	:	Specific discharge
$\underline{Q}$ (m^3/day/m)	:	Nodal discharges
$\underline{\underline{R}}$ (m/day)	:	Permeability matrix
S	:	Saturation
S_e	:	Effective saturation
S_{residu}	:	Residual saturation
S_{sat}	:	Saturation at $p = 0$
t (day)	:	Time
γ (kN/m^3)	:	Volumetric weight
ϕ (m)	:	Groundwater head
ϕ_p (m)	:	Pore pressure head